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Dynamics of a generalized Atwood's machine with three degrees of freedom

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We consider a generalized version of Atwood's machine (see [1]) when two bodies of masses m_1, m_2 ($m_2 \geq m_1$) are attached to opposite ends of a massless inextensible thread wound round two massless frictionless pulleys of negligibly small radius. Two separated pulleys are used to avoid collisions of the bodies. Body m_2 is constrained to move only along a vertical while body m_1 moves like a spherical pendulum of variable length. Such a system has three degrees of freedom and its motion is described by the following differential equations

$$\begin{aligned}(1+\mu)\ddot{r} &= r\dot{\theta}^2 - g(\mu - \cos\theta) + \frac{p_\varphi^2(1+\mu)^2}{r^3\sin^2\theta},\\ r\ddot{\theta} &= -2\dot{r}\dot{\theta} - g\sin\theta + \frac{p_\varphi^2(1+\mu)^2\cos\theta}{r^3\sin^3\theta},\\ \dot{\theta} &= \frac{p_\varphi(1+\mu)}{r^2\sin^2\theta}.\end{aligned} \tag{1}$$

Here r is a length of the thread between pulley and body m_1, φ and θ are the spherical angles, g is a gravitational constant, and parameter $\mu = m_2/m_1$. As there is no torque about the vertical line the system has an integral of motion $p_\varphi = r^2\dot{\theta}\sin^2\varphi/(1+\mu)$ that is determined from the initial conditions.

Note that in case of $p_\theta = 0$ body m_1 oscillates in a vertical plane and we obtain the swinging Atwood machine that was a subject of many papers (see, for example, [2], [3]). It was shown that even small oscillations can modify the system motion significantly and some unexpected kinds of motion such as periodic or quasi-periodic motion can arise.

Here we consider the case $p_\theta \neq 0$ when new kind of motion can arise. For example, there exists a conical motion when $r = r_0$, $\theta = \theta_0$ and $\dot{\varphi} = \omega$ are constants. The corresponding solution of system (1) describes a uniform motion of body m_1 in a horizontal plane on a circular orbit of radius $r_0 \sin\theta_0$. Simulation of the system motion shows that small variation of the initial conditions results only in small perturbation of the body m_1 orbit. Doing necessary symbolic calculation and analyzing the Hamiltonian function of the system we prove orbital stability of this solution. All relevant symbolic and numerical calculations and visualization of the results are performed with the computer algebra system Mathematica [4].

Keywords
Atwood's machine, Simulation, Periodic motion, Wolfram Mathematica

References

[1] G. Atwood, *A Treatisa on the Rectilinear Motion and Rotation of Bodies*. Cambridge University Press, 1784.
[2] N.B. Tufillaro, T.A. Abbott, D.J. Griffiths, Swinging Atwood's machine. *Amer. J. Phys.* **52**, 895–903 (1984).
[3] A.N. Prokopenya, Motion of a swinging Atwood's machine: simulation and analysis with Mathematica. *Mathematics in Computer Science* **11**, 417–425 (2017).
[4] S. Wolfram, *An elementary introduction to the Wolfram Language, 2nd ed.*. Champaign, IL, USA, Wolfram Media, 2017.