Applications of Computer Algebra – ACA 2019
Montréal, Canada | July 16-20, 2019
École de technologie supérieure

Reparameterizations and Lagrange piecewise-cubics for fitting reduced data

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The problem of estimating the unknown regular curve $\gamma : [0, T] \to \mathbb{E}^n$ from the so-called *reduced data* Q_m has been so far extensively studied in the related literature (see e.g. [1], [3] or [4]). In this setting, Q_m forms the collection of $m+1$ points $Q_m = \{q_i\}_{i=0}^m$ in arbitrary Euclidean space $\mathbb{E}^n$ satisfying the corresponding interpolation conditions $q_i = \gamma(t_i)$. Having selected a specific scheme $\hat{\gamma}$ to fit Q_m (see e.g. [1]), the unknown interpolation knots $\mathcal{T}_m = \{t_i\}_{i=0}^m$ obeying $t_i < t_{i+1}$ must be somehow compensated by their "estimates" $\hat{\mathcal{T}}_m = \{\hat{t}_i\}_{i=0}^m$ subject to $\hat{t}_i < \hat{t}_{i+1}$. Given Q_m, the appropriate choice of $\hat{\mathcal{T}}_m$ should guarantee potentially a fast convergence rate α in estimating γ by $\hat{\gamma}$ at best matching the underlying asymptotics in $\gamma \approx \hat{\gamma}$ as if the missing knots $\mathcal{T}$ were used. A possible recipe for $\hat{\mathcal{T}}_m \approx \mathcal{T}$ is to apply the so-called *exponential parameterization* $\hat{\mathcal{T}}_m^\lambda = \{\hat{t}_{i,\lambda}\}_{i=0}^m$ controlled by Q_m and a single parameter $\lambda \in [0, 1]$ - see e.g. [3]. A special case of $\lambda = 1$ yields a well-known *cumulative chord parameteriztion* discussed e.g. in [2], [3], [4] or [11]. The asymptotics in approximating γ by various $\hat{\gamma}$ based on $(Q_m, \hat{\mathcal{T}}_m^\lambda)$ are studied e.g. in [2], [4], [5], [6] or [7]. In particular, for *a modified Hermite interpolant* $\hat{\gamma} = \hat{\gamma}_H \in C^1$ (see [10]) and for an arbitrary $\gamma \in C^4([0, T])$ the following *sharp* result holds, uniformly over $[0, T]$ (see [4], [7] and [9]):

$$(\hat{\gamma}^H \circ \psi)(t) = \gamma(t) + O(\delta_m^1) \text{ for } \lambda \in [0, 1) \text{ and } (\hat{\gamma}^H \circ \psi)(t) = \gamma(t) + O(\delta_m^4) \text{ for } \lambda = 1, \quad (1)$$

where $\psi : [0, T] \to [0, \hat{T}]$ defined in [7] is implicitly parameterized by λ (here $\hat{T} = \hat{t}_{m,\lambda}$). Here $\delta_m = \min_{i \le 0 \le m-1}\{t_{i+1} - t_i\}$. The case of $\lambda \in [0, 1)$ requires to assume a thinner class of *more-or-less uniform samplings* (see [6]), whereas $\lambda = 1$ stipulates an admission of more general class of the so-called *admissible samplings* - see [4]. For certain applications ψ should constitute a genuine *reparameterization* (e.g. for length $d(\gamma)$ estimation by $d(\hat{\gamma})$). In other cases the mapping ψ needs to be *a non-injective mapping* (e.g. if extra loops in trajectory of $\hat{\gamma} \circ \psi$ are required). The last issue is recently studied for $\hat{\gamma}^H$ in [10]. An analogous asymptotics to (1) is established for Lagrange piecewise-cubics $\hat{\gamma} = \hat{\gamma}^C \in C^0$ in

[4], [8] and [11]. Here the mapping $\psi = \psi^c : [0, T] \to [0, \hat{T}]$, defines similarly a Lagrange piecewise-cubic satisfying $\psi^c(t_i) = \hat{t}_{i,\lambda}$.

In this work we formulate and prove sufficient conditions for ψ^c to yield $\dot{\psi}^c > 0$ for both *sparse and dense* reduced data Q_m. The latter enforces ψ^c to be *a reparameterization*. Geometrical and algebraic insight supported by illustrative visualization is also given with the aid of symbolic computations performed in *Mathematica* [12].

Keywords
Interpolation, Reduced data, Convergence, Sharpness and Parameterization

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