Syzygies for Translational Surfaces

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A translational surface is a rational tensor product surface generated from two rational space curves by translating either one of these curves parallel to itself in such a way that each of its points describes a curve that is a translation along the other curve. Translational surfaces, ruled surfaces, swept surfaces, along with low degree surfaces such as quadratic and cubic surfaces, are basic modeling surfaces that are widely used in computer aided geometric design and geometric modeling.

Since translational surfaces are generated from two space curves, translational surfaces have simple representations. The simplest and perhaps the most common representation of a translational surface is given by the rational parametric representation $\mathbf{h}^*(s;t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$, where $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$ are two rational space curves. Translational surfaces represented by $\mathbf{h}^*(s;t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$ have been investigated by differential geometers, and also studied from a geometric modeling point of view.

Translational surfaces defined by $\mathbf{h}^*(s;t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$ are not translation invariant: translating both curves $\mathbf{f}^*$ and $\mathbf{g}^*$ by the vector $\mathbf{v}$ translates the surface $\mathbf{h}^*$ by the vector $2\mathbf{v}$. One would like to define translational surfaces in such a way that translating the two generating curves by the same vector $\mathbf{v}$, also translates every point on the surface by the vector $\mathbf{v}$. In this presentation, we offer an alternative definition of translational surfaces given by the rational parametric representation $\mathbf{h}^*(s;t) = \frac{\mathbf{f}^*(s)+\mathbf{g}^*(t)}{2}$, where $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$ are two rational space curves. Under this definition, these translational surfaces consist of all the midpoints of all the lines joining a point on $\mathbf{f}^*$ to a point on $\mathbf{g}^*$, so these translational surfaces are invariant under rigid motions: translating and rotating the two generating curves translates and rotates these translational surfaces by the same amount. Hence, applying a rigid motion to a translational surface can be achieved by applying the same rigid motion to the two rational space curves that generate the surface. Therefore, one can control these translational surfaces simply by manipulating the generating curves.

In this presentation, we will investigate the translational surfaces given by the rational parametric representation $\mathbf{h}^*(s;t) = \frac{\mathbf{f}^*(s)+\mathbf{g}^*(t)}{2}$. Our main goal is to utilize syzygies to study translational surfaces. We will construct three special syzygies for a translational surface from the μ-basis of one of the generating space curves. In addition, we will examine many properties of translational surfaces, and compute the implicit equation and singularities from these three special syzygies.

The outline of the presentation is structured as the following. First, we introduce the definition of translational surfaces, provide a few examples of translational surfaces generated from two rational space curves, and investigate a few special characteristics of translational surfaces. Second, we study syzygies of translational surfaces, relate the syzygies of the generating curves to the syzygies of the corresponding translational surface, and compute the implicit equation of a translational surface from the resultant of the three moving planes. Third, we focus on ruled translational surfaces and compute their implicit equations based solely on the μ-bases of the generating curves. Fourth, we detect the self-intersections of translational surfaces. Finally, we observe that the techniques used in this paper can be applied with only minor modifications to the translational surfaces defined by $\mathbf{h}^*(s;t) = a\mathbf{f}^*(s) + b\mathbf{g}^*(t)$, where a, b are real numbers and $ab \neq 0$. In the case of $a = b = 1$, we provide a necessary and sufficient condition for a rational tensor product surface to be a translational surface.

Systems of polynomial equations arise throughout mathematics, science, and engineering. Algebraic geometry provides powerful theoretical techniques for studying the qualitative and quantitative features of their solution sets. This talk presents algorithmic tools for algebraic geometry and experimental applications, as well as introduces software systems in which the tools have been implemented and with which the experiments can be carried out. Computer algebra system such as Singular [1], Macaulay 2 [2], Maple [3], and Mathematica [4] are used to compute examples and generate graphics.

For instance, consider the translational surface given by

$$\mathbf{h}^*(s;t) = \frac{(s^2-1, s(s^2-1), 0)}{2} + \frac{(t, 0, -t^2)}{2} = \frac{\mathbf{f}^*(s) + \mathbf{g}^*(t)}{2}. \quad (1)$$

Figure 1 generated by Mathematica [4] is an affine view of the surface $\mathbf{h}^*(s;t)$ given in Equation (1), where the highlighted curves are the curves $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$.

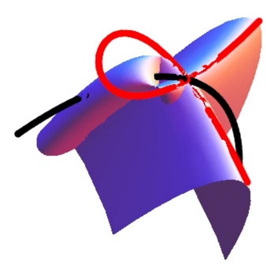

Figure 1: Surface $\mathbf{h}^*(s;t) = \frac{(t+(s^2-1), s(s^2-1), -t^2)}{2}$

The translational surface given by Equation (1) has a base point. The search for techniques for implicitizing rational surfaces with base points is a very active area of research because base points show up quite frequently in practical industrial design. It is often difficult to compute the multiplicity of base points, and to implicitize a surface that has a complicated collection of base points. Singular [1] and Macaulay 2 [2] have computer algebra packages aimed at algebraic geometry and commutative algebra to compute the multiplicity of the base points. The implicit equation of the surface $\mathbf{h}^*(s;t)$ in Equation (2) is computed from the resultant of three moving planes. Maple [3], Singular [1], and Macaulay 2 [2] have implemented packages to compute multivariate resultant. We carried out our computation via Macaulay 2 [2].

$$\begin{aligned} F(x,y,z) &= 4x^4+16x^5+16x^6-8x^2y^2-16x^3y^2+4y^4+4x^2z+16x^3z \\ &\quad +24x^4z+4y^2z+24xy^2z+z^2+4xz^2+12x^2z^2+2z^3 \\ &= 0. \end{aligned} \tag{2}$$

References

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[3] Maple 2016. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.

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