A Modified Hermite Interpolation with Exponential Parameterization

R. Kozera[1,2] and M. Wilkołazka[3]

[1] *Warsaw University of Life Sciences - SGGW, Poland, ryszard.kozera@gmail.com*
[2] *The University of Western Australia, Perth, Australia,ryszard.kozera@csse.uwa.edu.au*
[3] *The John Paul II Catholic University of Lublin, Lublin, Poland,magdalena.wilkolazka@gmail.com*

This work addresses the problem of estimating the unknown trajectory of a regular curve $\gamma : [0,T] \to E^n$ based on the so-called *reduced data* Q_m. The latter represent $m+1$ ordered interpolation points $Q_m = \{q_i\}_{i=0}^m$ (with $q_{i+1} \neq q_i$) in arbitrary Euclidean space E^n subject to the constraint $q_i = \gamma(t_i)$. We assume that the respective knots $\mathscr{T}_m = \{t_i\}_{i=0}^m$ satisfying $t_i < t_{i+1}$ are not given. In order to fit Q_m with the prescribed interpolation scheme, one also needs to substitute somehow the unknown knots $\mathscr{T}_m$ with another family of parameters $\hat{\mathscr{T}}_m = \{\hat{t}_i\}_{i=0}^m$ satisfying $\hat{t}_i < \hat{t}_{i+1}$. In doing so, the so-called *exponential parameterization* depending on a single parameter $\lambda \in [0,1]$ and Q_m can e.g. be used. This ultimately yields $\hat{\mathscr{T}}_m^\lambda = \{\hat{t}_i^\lambda\}_{i=0}^m \approx \mathscr{T}_m$ - see e.g. Refs. [1, 2]. Note that a special case of $\lambda = 1$ introduces the so-called *cumulative chord parameterization* of reduced data Q_m (see e.g. Ref. [1]). In the next step a classical *Hermite interpolation* (see Ref. [3]) $\hat{\gamma}_H : [0,\hat{T}] \to E^n$ based on Q_m and $\hat{\mathscr{T}}_m^\lambda$ can be invoked (with $\hat{T} = \hat{t}_m^\lambda$). However, the respective missing velocities $\{v_i = \dot{\gamma}(t_i)\}_{i=0}^m$ along Q_m are approximated here according to $\hat{v}_i = \hat{\gamma}_{3,i}^{'}(\hat{t}_i^\lambda)$, where $\hat{\gamma}_{3,i} : [\hat{t}_i^\lambda, \hat{t}_{i+3}^\lambda] \to E^n$ denotes a standard Lagrange cubic satisfying $\hat{\gamma}_{3,i}(\hat{t}_{i+j}) = q_{i+j}$ (for $j = 0,1,2,3$) - see Ref. [3]. Note that here we apply in fact "overlapped" Lagrange cubics to estimate all velocities $\{v_i\}_{i=0}^m$ at $\{Q_m\}_{i=0}^m$. More precisely, for $\hat{\gamma}_{3,i+1} : [\hat{t}_{i+1}^\lambda, \hat{t}_{i+4}^\lambda]$ interpolating $\{q_{i+1+j}\}_{j=0}^3$ we adopt a similar estimate i.e. $\hat{v}_{i+1} = \hat{\gamma}_{3,i+1}^{'}(\hat{t}_{i+1}^\lambda)$ of $\dot{\gamma}(t_{i+1})$. For the last four interpolation points $\{q_i\}_{i=m-3}^m$ the above procedure can be repeated upon changing the order of points and taking the computed derivatives with the opposite sign. Such construction of $\hat{\gamma}_H$ based on $\hat{\mathscr{T}}_m^\lambda$, $\{\hat{v}_i\}_{i=0}^m$ and Q_m is coined *a modified Hermite interpolation*. A special case when $\lambda = 1$ is discussed in more details in Refs. [4, 5].

Given $\delta_m = \max_{0 \leq i \leq m-1}\{t_{i+1} - t_i\}$ the sampling $\mathscr{T}_m$ is called *admissible* if $\lim_{m\to\infty} \delta_m = 0$. The subfamily of admissible samplings is called *more-or-less uniform* if there exists $\beta \in (0,1]$ such that $\delta_m \beta \leq t_{i+1} - t_i$, holding for all $i = 0,1,\ldots,m-1$ and arbitrary m. The question of approximating γ by modified Hermite interpolant $\hat{\gamma}_H$ is studied merely for the special case of $\lambda = 1$ i.e. for cumulative chord parameterization in Refs. [4, 5]. More specifically, quratic order of convergence in trajectory approximation is proved and confirmed numerically in the above last cited papers. We extend this result to the remaining $\lambda \in [0,1)$

determining the exponential parameterization. Indeed the following holds:

Theorem 1 *Assume that a regular* $\gamma : [0,T] \to E^n$ *of class* C^4 *with the unknown interpolation knots* $\{t_i\}_{i=1}^m$ *is sampled more-or-less uniformly. If* $\hat{\gamma}_H$ *represents a modified Hermite interpolant based on reduced data* Q_m *and exponential parameterization governed by* $\lambda \in [0,1]$*, then for some piecewise-cubic-*C^∞ $\psi : [0,T] \to [0,\hat{T}]$*:*

$$(\hat{\gamma}_H \circ \psi)(t) = \gamma(t) + O(\delta_m^1) \;\; for \;\; \lambda \in [0,1) \; and \; (\hat{\gamma}_H \circ \psi)(t) = \gamma(t) + O(\delta_m^4) \;\; for \;\; \lambda = 1. \tag{1}$$

Theorem 1 establishes a substantial deceleration in convergence rates for trajectory estimation (to the linear one) while λ runs over $[0,1)$. The latter contrasts with the fast quartic order holding for $\lambda = 1$ as specified in (1) (see also Ref. [4]). The numerical tests conducted in this work (with the aid of *Mathematica* package - see Ref. [7]) confirm the sharpness of the estimates from (1). A similar effect of the left-hand side discontinuity in convergence rate at $\lambda = 1$ is proved for piecewise-quadratic Lagrange intepolation based on exponential parameterization and Q_m - see Refs. [2, 6]. Fitting reduced data is an important problem in computer vision and graphics, as well as in engineering, microbiology, physics and other applications like medical image processing - see e.g. [1].

References

[1] B.I. Kvasov, *Methods of Shape-Preserving Spline Approximation*, World Scientific Publishing Company, Singapore (2000).

[2] R. Kozera and L. Noakes, *Piecewise-quadratics and exponential parameterization for reduced data*, Applied Mathematics and Computation **221**, pp. 1–19 (2013).

[3] C. de Boor, *A Practical Guide to Spline*, Springer-Verlag, New York Heidelberg Berlin (1985).

[4] R. Kozera and L. Noakes, C^1 *interpolation with cumulative chord cubics*, Fundamenta Informaticae **61**, 3-4, pp. 285–301 (2004).

[5] R. Kozera, *Curve modeling via interpolation based on multidimensional reduced data*, Studia Informatica **25**, 4B(61), pp. 1–140 (2004).

[6] R. Kozera and L. Noakes, *Piecewise-quadratics and* ε*-uniformly sampled reduced data*, Applied Mathematics and Information Sciences **10**, 1, pp. 33–48 (2016).

[7] S. Wolfram, *The Mathematica Book*, Wolfram Media, 5th. ed. (2003).