On degenerate central configurations in the N-body problem

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The existence of central configurations in the N-body problem is connected with two type of parameters, masses of the bodies, which are physical parameters, and positions of the bodies as geometrical parameters. The main goal concerning central configurations is to determine the values (m_i, q_i), $i = 1, \ldots, N$ of the parameters for which such arrangement of the N point masses exists. The well known list of classical central configurations of Euler and Lagrange has been completed by B. Elmabsout [1]. He has added a configuration consisting of $2n$ equal point masses, located at the vertices of two regular concentric n-gons, while the point mass with nonzero mass m_0 lies in the center of these polygons. He has proved that such configuration exists if and only if these two polygons are homothetic, or differ by an angle of π/n.

The bifurcation's problem is the second one which is connected with the degenerate central configurations and for which lots of interesting questions arise. The authors which have dealed with bifurcations in the n-body problem are M.Sekiguchi [4] and J.Lei, M.Santoprete [5]. They have analyzed a highly symmetrical *rosette configuration* of $2n+1$ point masses; Sekiguchi has considered $2n$ point masses with mass m situated at the vertices of two different coplanar and concentric regular n-gons, whilst Lei, Santoprete have analized $2n$ point masses, for which n particles with m_1 mass are located at the vertices of n-gon and n particles with the mass m_2 lie at the vertices of another n-gon. Both n-gons are regular, concentric and rotated of an angle $\frac{\pi}{n}$. The mass m_0 lies at the center of masses. Sekiguchi has established that if $n \geq 3$ and $\mu < \mu_c$, where the parameter μ controls bifurcations, then there exist three central configurations, otherwise only one for $\mu \geq \mu_c$. J.Lei and M.Santoprete have shown that if $n \geq 3$ then there exists a degenerate central configuration and a bifurcation for value of the parameters μ, ε.

Assume that $m_j \in R^+$ $(j = 1, ..., N)$ and $q \neq q'$ are distinct radii of two concentric regular polygons. We deal with a new class of central configurations in the N-body problem. By $(2, N)$ we understand the configuration which consists of n point masses $m_1, m_2, \ldots, m_n$ situated at the vertices of the regular n-gon, and of $2n$ point masses $m_1^{'}, m_2^{'}, \ldots, m_{2n}^{'}$ located at the vertices of the regular $(2n)$-gon, so, $N = 3n$.

The existence of such configuration has been shown numericaly by E.A.

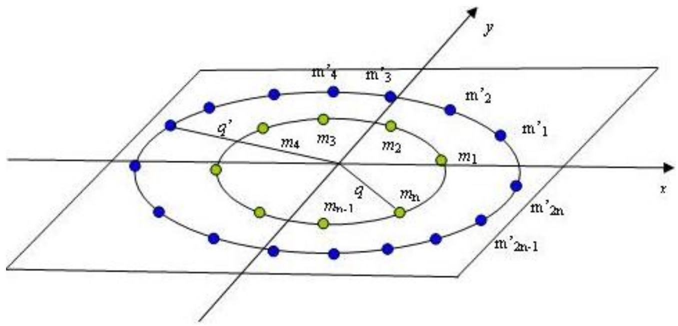

Figure 1. Configuration $(2,N)$.

Grebenikov [2] and has been established in the case $n=2$ by A. Siluszyk [3]. In our work we deal with a family of central configurations of n point masses, when $n=2$. If there exists a constant λ such that [6]

$$m^{-1}\frac{\partial U}{\partial \rho} = \lambda \rho, \tag{1}$$

then the configuration $\rho = (q_1, q_2, \ldots, q_N) \in \{R^2 \setminus \Delta\}$ is called a *central configuration* in the N-body problem; here $U = \sum_{1 \le i<j \le N} \frac{m_i m_j}{r_{ij}}$, $r_{ij} = |q_j - q_i|$, $\Delta = \{\rho : q_i = q_j, i \neq j\}$ and $m = diag\,[m_1, m_2, \ldots, m_N]$. Any central configuration gives *relative equilibrium* which rotates in the plane with constant angular velocity ω and remains congruent to itself for all times. We assume that $\sum_{i=1}^{N} m_i q_i = 0$, i.e., the center of mass of our system is located at the origin, moreover, we know [6] that $\lambda = \frac{U(\rho)}{I(\rho)}$, where $I(\rho) = \rho^T m \rho$ means the moment of inertia. The set of equivalence classes of central configurations are determined by critical points of $U(\rho)|_X$, where $X = \{R^2 \setminus \Delta : I(\rho) = 1\}$, $I = \sum_{i=1}^{N} m_i q_i^2$ [7]. Moreover the critical points which are degenerated, i.e. the Hessian $D^2(U(\rho)|_X)$ at such points has a nontrivial nullspace, give us an equivalence class of degenerated central configurations.

We deal with configuration of the type $(2,N)$ with any natural N and $m_1, m_2, m_3 > 0$, $q, q' > 0$. Then we have:

Theorem 1. ([3]) For any natural N and any real numbers $m_1 > 0$, and q, $r > 0$ ($r = \frac{q'}{q}$) a central configuration of type $(2,N)$ exists if and only if

$$\begin{cases} m_2 = \frac{r^2}{(\kappa_1-\kappa_6)(\kappa_1-4r^3(\kappa_2+\kappa_4)+4r^4(\kappa_3+\kappa_5)+\kappa_6)}(-16r^5(\kappa_2-\kappa_4)\kappa_5+ \\ \quad +16r^4(\kappa_2\kappa_4-\kappa_4^2+\kappa_5(\kappa_3-\kappa_5))-16r^3\kappa_4(\kappa_3-\kappa_5)+ \\ \quad -r(-\kappa_1^2-4\kappa_4\kappa_6+\kappa_1(4\kappa_2+\kappa_6))+4\kappa_1\kappa_3-4\kappa_5\kappa_6)m_1, \\ m_3 = \frac{r^2}{(\kappa_1-\kappa_6)(\kappa_1-4r^3(\kappa_2+\kappa_4)+4r^4(\kappa_3+\kappa_5)+\kappa_6)}(16r^5\kappa_3(\kappa_2-\kappa_4)+ \\ \quad -16r^4(\kappa 2(\kappa 2-\kappa_4)+\kappa_3(\kappa_3-\kappa_5))+16r^3\kappa_2(\kappa_3-\kappa_5)+ \\ \quad +r(\kappa_1(\kappa_1-4\kappa_4-\kappa_6)+4\kappa_2\kappa_6)+4(\kappa_1\kappa_5-\kappa_3\kappa_6))m_1, \end{cases} \tag{2}$$

(here, the sums in the right hand side do not depend on k). The elements κ_s, $s=1,..,6$ for any $k=1,\ldots,n$ are expressed by [3]:

$$\left\{\begin{array}{lcl}
\kappa_1 & = & \sum\limits_{j=1}^{n} \frac{1}{\left|\sin\frac{\pi(k-j)}{n}\right|}, \\
\kappa_2 & = & \sum\limits_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(\frac{2\pi(j-k)}{n}+\frac{\pi}{n}))^{\frac{3}{2}}}, \\
\kappa_3 & = & \sum\limits_{j=1}^{n} \frac{\cos(\frac{2\pi(j-k)}{n}+\frac{\pi}{n})}{(1+r^2-2r\cos(\frac{2\pi(j-k)}{n}+\frac{\pi}{n}))^{\frac{3}{2}}}, \\
\kappa_4 & = & \sum\limits_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(\frac{2\pi(j-k)}{n}+\frac{2\pi}{n}))^{\frac{3}{2}}}, \\
\kappa_5 & = & \sum\limits_{j=1}^{n} \frac{\cos(\frac{2\pi(j-k)}{n}+\frac{2\pi}{n})}{(1+r^2-2r\cos(\frac{2\pi(j-k)}{n}+\frac{2\pi}{n}))^{\frac{3}{2}}}, \\
\kappa_6 & = & \sum\limits_{j=1}^{n} \frac{1}{\left|\sin(\frac{2\pi(j-k)}{2n}+\frac{\pi}{2n})\right|}.
\end{array}\right. \tag{3}$$

Here, we consider the relative equilibrium of six bodies; such configuration consists of (m_i,q_i), $i=1,...,6$ point masses, two of them with equal masses m_1 situated at the ends of a segment having lenght $2q$, while other two pairs of masses m_2 and m_3 are located at the vertices of a square whose side is $q'\sqrt{2}$. Moreover, the vertices of this square are on the axises of symmetry of the segment. The Newtonian potential and the moment of inertia for the configuration of the type $(2,6)$ are given by:

$$\left\{\begin{array}{lcl}
U & = & \frac{m_1^2}{q}((\frac{1}{2}+\frac{\mu^2+\nu^2}{2r})+4\frac{\mu}{\sqrt{1+r^2}}+2\nu(\frac{1}{\sqrt{(r-1)^2}}+\frac{1}{\sqrt{(r+1)^2}})+2\sqrt{2}\frac{\mu\,\nu}{r}), \\
I & = & 2m_1q^2(1+r^2(\mu+\nu)).
\end{array}\right. \tag{4}$$

for parameters $\mu=\frac{m_2}{m_1}>0$ and $\nu=\frac{m_3}{m_1}>0$, $r=\frac{q'}{q}$.

Theorem 2. Suppose, that $(2,6)$ is configuration of six bodies with $q,q'>0$ as mutual distances corresponding to $|Om_1|$, $|Om_2|=|Om_3|$, respectively. Then for all $m_1,m_2,m_3>0$ there exists at least one central configuration of the type $(2,6)$.

Solving the equation $\nabla U(\rho)+\frac{1}{2}\sigma\nabla I(\rho)=0$ with respect to unknown q,r from (4), we obtain in the plane $I=1$ the equation $U'_r\cdot I'_q-U'_q\cdot I'_r=0$. The solutions of this equation give us all central configurations of the $(2,6)$.

Proposition 1. For any $m_1>0$ there exists at least one r^* for which the configuration of the type $(2,6)$ is the equivalence class of a degenerated central configuration.

Computer Assisted Proof methods are applied to obtain the main results of our report. These methods represent a new technique for obtaining rigorous results concerning the global dynamics of nonlinear systems of the equations (see, e.g., P. Zgliczynski [8]). Below we present the number of critical points for r^* and critical masses μ^*, ν^* (here, we put $m_1 = 0.001$):

r^*	critical masses μ^*, ν^*	number of c.p.
0.70582611533?	$\mu^* = 58.38984663223665?$, $\nu^* = 0.944757994652664?$	8
0.6226157?	$\mu^* = 30.07078175550046?$, $\nu^* = 2.359914736266285?$	12
0.5025690268698576?	$\mu^* = 3.18858 * 10^{13}$, $\nu^* = 3.18858 * 10^{13}$	14
0.5646596968686869?	$\mu^* = 22.45513293937953?$, $\nu^* = 5.061034347849054?$	16
9.3445?	$\mu^* = 15.117653893458127?$, $\nu^* = 0.017734428734047708?$	20

Table 1: Number of critical points for given distances r^* and masses μ^*, ν^*.

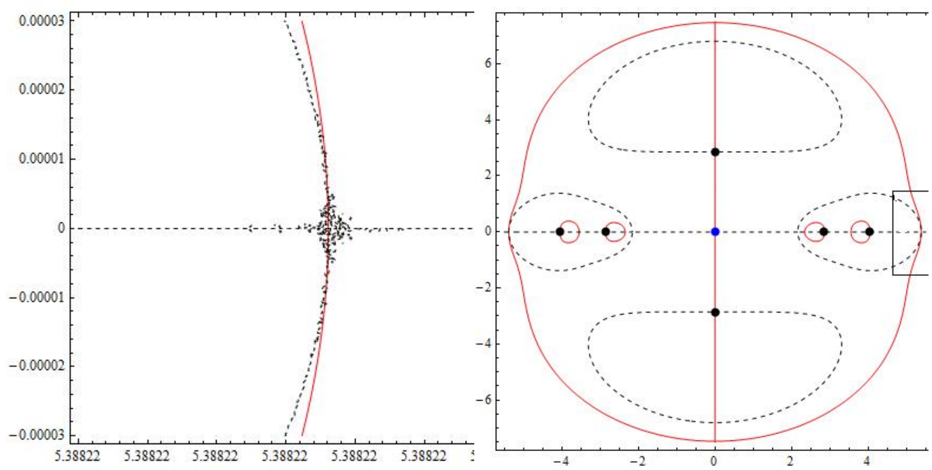

Figure 2. The degenerate central configuration for $r^* = 0.70582611533?$, $\mu^* = 58.38984663223665?$ and $\nu^* = 0.944757994652664?$.

The change which is presented by Table 1 and Figure 2 expresses an exchange of the number of solutions 16 for 8, 12, 14 and 20 in the equation $U'_r \cdot I'_q - U'_q \cdot I'_r = 0$.

In our studies we apply a few theoretical facts concerning interval arithmetic. Using a theorem and some methods it is possible to find rigorous bounds on roots of nonlinear equation. We use Sage arbitrary precision real intervals, in which an interval $[a,b]$ is written as a standard floating-point number with a question mark, where question mark can be interpreted as an error.

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