Application of Computer Algebra System and the Mean-Value Theory for Evaluating Electrostatic Potential and its Associated Field for Nontrivial Configurations

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Evaluation of electrostatic potential at an arbitrary point within a two dimensional region free of electric charge containing geometrically dispersed nontrivial configurations electrified to constant potentials applying the standard classic approach, i.e. Laplace equation is challenging. The challenge stems from the fact that the solution of the Laplace equation needs to be adjusted to the boundary conditions imposed by the configurations. Numeric solution of the latter is challenging as it lacks generalities. An entirely different numeric solution method is based on the application of The Mean-Value Theory. The latter is a pure numeric approach; although the output of its iterated refined version is successful, it is cumbersome. In this investigation utilizing the powerful features of Computer Algebra Systems (CAS), specifically *Mathematica* by a way of example we show an innovative approach. Our approach is based on a combination of numeric aspect of The Mean-Value Theory on one hand and *Mathematica* features on the other hand. This semi numeric-symbolic approach not only provides the desired output, but it also generates information beyond the scope of the standard classic method. By way of example we present the intricacies of our approach, showing 1) how the potential is evaluated and 2) how corollary information not addressed in classic cases such as electric field is calculated as well. Our method is applied to a two-dimensional case; its three dimensional version may easily be applied to cases of interest.

1 Motivations and Goals

In two dimensional electrostatic it is a classic practice to map the potential that arises from a single common geometric object such as a line, a square, a circle and etc that is electrified to a potential. Stepping away from these cases one encounters multiple-body geometric configurations, each charged to a certain potential. Addressing the latter not only theoretically is interesting but is valued for practical applications. Analytic solutions of these scenarios mathematically are challenging and because each scenario embodies a specific configuration, solutions lack the generalities. The mathematical challenges stem from the fact that the potential, ϕ that is subject to Laplace equation, $\nabla^2\phi = 0$ ought to be in compliance with the boundary conditions imposed by the geometry of the configuration. As such, in most cases one relies on the numeric solution of the Laplace equation [1],[2],[3]. An entirely alternative approach to addressing the same issue is a pure numeric method of another sort. This method by-passes the Laplace equation in its entirety; it is called The Mean-Value Theory, see for instance its applications [4]. Generally speaking one drops a virtual fishnet on the given configuration dividing the region of interest in grids. To begin with one assigns a wisely chosen guesstimated numeric potential to each node of the grid. Then one replaces the initial nodal potentials with the average of the potentials of the closest nodes. Repetition of the procedure stabilizes the potentials. The accuracy of the output is controlled by 1) recursive repetitions and 2) the fishnet mesh size; the smaller the mesh the better the output. This method appears to require either cumbersome manual or programming efforts.

Being aware of the latter issues, we present an effective, a short-cut approach curing both aforementioned challenges. The core of the solution is based on utilizing a Computer Algebra System (CAS) specifically *Mathematica* [5],[6]. To demonstrate the approach we craft our investigation that is composed of three sections. In addition to Motivations and Goals, in Section 2 by a way of example we present the detailed analysis. This section also includes the results and associated graphic output. Having this information on hand we further the analysis by evaluating the electric field. This is a fresh idea, literature lacks this information. We close our work with a few remarks.

2 Physics of the problem and its solution

Consider a set of two two-dimensional kinked metallic structure shown in Fig 1. The segments symmetrically are separated with a gap, and horizontally are extended to infinity. Assume the bottom and the top pieces are electrified to constant potentials e.g., $\phi = 0$ and $\phi = 3.0V$, respectively. The given structure resembles the profile of an unusual parallel-plate capacitor; this structure is suggested in [7]. It is one of the objectives of this investigation to determine the electrostatic potential at any point within the plates.

According to what is outlined in Sect. 1, in order to evaluate the potential we drop a fishnet with a coarse mesh size on the region of interest. This is shown in

Fig 1.

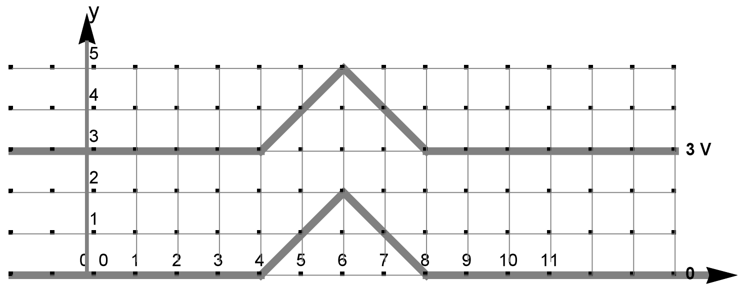

Figure 1. Display of the kinked structure with its accompanied fishnet grid.

The origin of the coordinate system conveniently is set as shown. The fishnet is composed of a 5*x*11 grid. The node at the top left is p[1,1] and the one at the bottom right is p[6,11]. The matrix below is the intuitive nodal start-up potential assignment of the entire grid; potential of the exterior nodes are designated by x.

$$\begin{pmatrix} x & x & x & x & x & 3. & x & x & x & x & x \\ x & x & x & x & 3. & 2. & 3. & x & x & x & x \\ 3. & 3. & 3. & 3. & 2. & 1. & 2. & 3. & 3. & 3. & 3. \\ 2. & 2. & 2. & 2. & 1. & 0. & 1. & 2. & 2. & 2. & 2. \\ 1. & 1. & 1. & 1. & 0. & x & 0. & 1. & 1. & 1. & 1. \\ 0. & 0. & 0. & 0. & x & x & x & 0. & 0. & 0. & 0. \end{pmatrix}$$

Utilizing this input we apply the nearest averaging values; for the scenario at hand only four values would contribute. In other words one fourth of the sum of the four closest adjacent nodal potentials of the start up values replaces the initial chosen node's potential. Repetition of the procedure stabilizes the potentials. The presentation version displays two matrices after one and four repetitions, respectively.

As shown, the difference between the chosen nodal potentials for the scenario at hand just after only four recursions are negligibly small; i.e. potentials are stabilized. Customarily, for two distinct reasons 1) to achieve a higher numeric accuracy and 2) more importantly, for determining the potential at any point one is to refine the mesh size. This is straightforward; however, it is cumbersome. Instead we devised a fresh, innovative approach. We utilize *Mathematica* numeric interpolation. This operation utilizes the stabilized nodal potentials and in one step produces a refined output as if an extremely fine mesh is used. What follows is the numeric and accompanied graphic output of the interpolated procedure.

y	V(Volt)
2.	0.
2.2	0.426844
2.4	0.813969
2.6	1.16306
2.8	1.47581
3.	1.75391
3.2	1.99903
3.4	2.21288
3.6	2.39713
3.8	2.55347
4.	2.68359
4.2	2.78919
4.4	2.87194
4.6	2.93353
4.8	2.97566
5.	3.

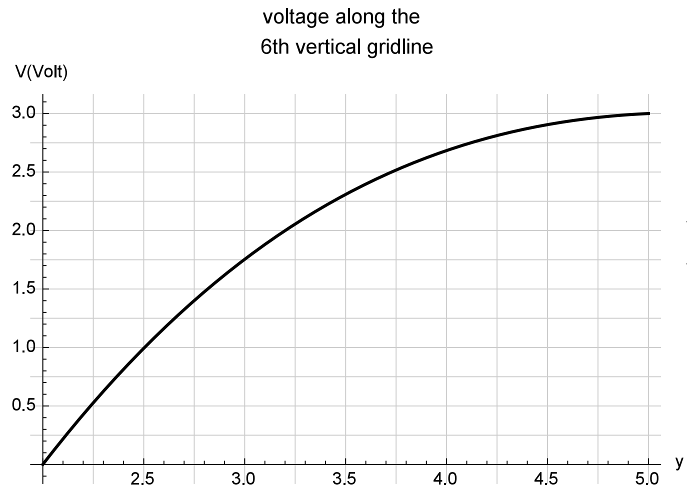

Figure 2. The first column of the table is the ordinates of the nodes. The second column is the corresponding potentials. The graph is the display of the adjacent table.

As shown the first column of the table includes not only the discrete integer y-values of the nodes but ordinates of the points between the nodes. Consequently, as explained in the text utilizing interpolation the second column embodies the corresponding potentials. The adjacent graph is the display of the Table. Accordingly, the interpolated plot gives the potential of any arbitrary y-valued ordinate. For the sake of clarification the output of this procedure is detailed for the 6th vertical grid-line. One may follow the same approach tabulating and plotting curves for any of the vertical grid-lines. Next we extend the procedure for horizontal grid lines with ordinates of 1,2 and 3. These are shown in Fig 3. Each set of curves is composed of a pair of lines. The dashed lines represents the point-to-point connected curves; the smooth solid lines are the interpolated curves.

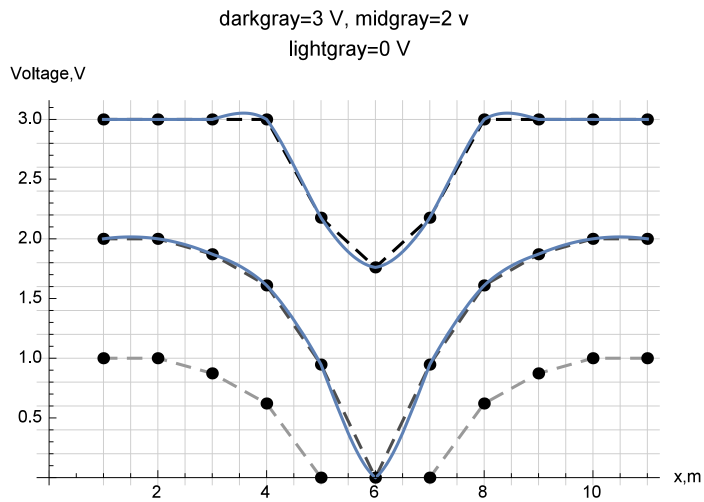

Figure 3. Coordinate of the x-axis is the same as the coordinate of the grid-lines in Fig 1. Dots are the potentials corresponding to the last matrix in the text. The dashed lines are the point-to-point connected curves. The smooth solid curves are the interpolated potentials.

With this information at hand utilizing, $\vec{E} = -\vec{\nabla}\phi$ surprisingly we are able to calculate the associated electric fields. This procedure embodies two valued points: 1) as shown the interpolated data is a continuous function making the gradient operable. Otherwise we would have to replace the gradient with a difference equation i.e., $\frac{\Delta\phi}{\Delta\xi}$where ξ is the distance between the potential contours. This would have given less accurate fields. 2) As shown in Fig 3 potential is a two dimensional function, $\phi(x,y)$ so that its associated field is a two dimensional vector, $\{E_x, E_y\}$. For a sake of completeness two such continuous fields associated with two grid-lines are tabulated and graphed. Presentation version displays the tables and their associated figures.

By tabulating and plotting these fields we illustrate that with a coarse mesh shown in Fig 1 we are able to evaluate the fields as if the mesh was refined and optimized.

3 Conclusions

Obtaining analytic solution even for two dimensional semi complicated geometrically dispersed electrified objects is challenging. Laplace equation is the master equation that needs to be adjusted to the relevant boundaries; this makes the solution peculiar to a specific scenario, as such it lacks generalities. An alternative solution other than numeric solution of Laplace equation is the Mean-Value Theory. This requires refined cumbersome programming. In this investigation we show utilizing a Computer Algebra System (CAS), specifically *Mathematica* a less cumbersome, satisfactory shortcut solution can be obtained. By way of example we present the specifics of our approach. For the sake of completeness we utilize the numeric output of the analysis and semi-analytically computed axillary quantities such as the fields. The presented approach conveniently may be applied to configurations of interest and readily may be extended to 3D configurations.

References

[1] Gilberto Urroz, Numerical Solution of Laplace Equation, ocw.usu.edu/LaplaceDirichletTemperature.pdf

[2] B. Hansen, Numerical Solution of Laplace's Equation, Syracuse University, New York 13244, 1992.

[3] Nunerical Solutions to Laplace's Equation, PHY220C (Electromagnetic Fields), newton.ex.ac.uk/teaching/CDHW/EM/CW960313-2.pdf

[4] B.A. Kemp, et al, Application of the electrostatic mean-value theorem to electrostatic sensor electrodes, Journal of Electrostatic - 65 (2007) 69-74, Elsevier.

[5] Mathematica^{TM} is symbolic computation software, V10.3, Wolfram Research Inc., 2015.

[6] Stephen Wolfram, "Mathematica book", 3rd ed., Cambridge University Press, 1996.

[7] Hans Ohanian, "Physics", 2nd ed. Expanded, W.W. Norton & Company, Inc., 500Fifth Ave., New York, 1989.