Motion of a Swinging Atwood's Machine: Simulation and Analysis

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The Atwood machine is a well-known device that is usually used to demonstrate the uniformly accelerated motion (see Ref. [1]). It consists of two bodies attached to opposite ends of a massless inextensible thread wound round a massless frictionless pulley. It is assumed that each body can move only along a vertical, and the thread doesn't slip on the pulley. Such Atwood's machine is a simple mechanical system with one degree freedom and one can easily show that the bodies acceleration is given by

$$a = \frac{m_2 - m_1}{m_2 + m_1}\, g\ , \tag{1}$$

where m_1 and m_2 $(m_2 > m_1)$ are the bodies masses, and g is the gravity acceleration. As the acceleration a can be easily found experimentally Eq. (1) is often used to find the local Earth gravity constant g.

However, doing the corresponding experiment, one can observe that the result obtained may differ noticeably from the true value of g. Trying to explain such result, one can notice that it is very difficult to constrain the bodies to move strictly along verticals and to avoid their oscillations.

It should be noted that swinging Atwood's machine has been a subject of a number of papers (see, for example, Refs. [2, 3, 4, 5, 6]) and its mechanical behaviour has been studied quite well. In particular, it has been proven that the system of differential equations describing dynamics of swinging Atwood's machine is not integrable, in general. It has been shown also that, depending on the mass ratio m_2/m_1, the system can demonstrate different types of motion, namely, periodic, quasi-periodic, or chaotic motion but physical reasons of such behaviour of the system and significance of oscillations has not been usually discussed.

The main purpose of the present paper is to analyze the terms appearing in the equations of motion owing to oscillations and to study their influence on the system behaviour. Combining symbolic and numerical calculations turns out to be very useful for this study because the equations of motion are not integrable. The validity of the results obtained is demonstrated by means of the simulation of motion of the swinging Atwood machine with the computer algebra system Mathematica (see Ref. [7]) that is used for doing all relevant symbolic and numerical calculations and visualization of the results.

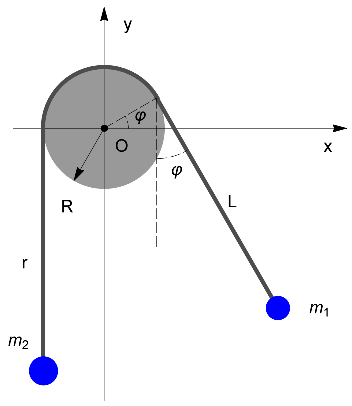

Figure 1: The swinging Atwood machine.

1 Equations of Motion

We consider here a generalized model of the simple Atwood machine when the body of mass m_1 is allowed to swing in a plane while the other body of mass m_2 is constrained to move along a vertical (see Fig. 1). Such a system has two degrees of freedom and its geometrical configuration can be described in terms of two variables, for example, an angle of the pulley rotation ψ, and an angle φ determining deviation of the thread from a vertical. Note that the length L of the thread between the body m_1 and the point, where the thread departs from pulley, is given by the relationship $L = L_0 + R(\varphi - \psi)$, where L_0 is its initial value, R is a radius of the pulley, and initial values of ψ and φ are assumed to be equal ($\psi_0 = \varphi_0$). Assuming the thread doesn't slip on the pulley, we obtain $r = r_0 + R(\psi - \psi_0)$, where r_0 is an initial length of the thread between the body m_2 and the pulley.

The Lagrangian of the system can be written in the form

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{2} m_1 \left(L_0 + R(\varphi - \psi)\right)^2 \dot{\varphi}^2 + \frac{1}{2}\left(I_0 + (m_1 + m_2)R^2\right)\dot{\psi}^2 \\ & - m_1 g \left(R \sin\varphi - (L_0 + R(\varphi - \psi))\cos\varphi\right) + m_2 g R \psi \ , \end{aligned} \tag{2}$$

where the dot denotes differentiation with respect to time, and I_0 is a moment of inertia of the pulley. Using Eq. (2) and doing standard symbolic calculations, we obtain the equations of motion in the form

$$\ddot{\psi} = \frac{R}{I_0 + (m_1 + m_2)R^2}\left(g(m_2 - m_1\cos\varphi) - m_1(L_0 + R(\varphi - \psi))\dot{\varphi}^2\right) \ , \tag{3}$$

$$\ddot{\varphi} = \frac{1}{L_0 + R(\varphi - \psi)}\left(2R\dot{\psi}\dot{\varphi} - g\sin\varphi - R\dot{\varphi}^2\right) \ . \tag{4}$$

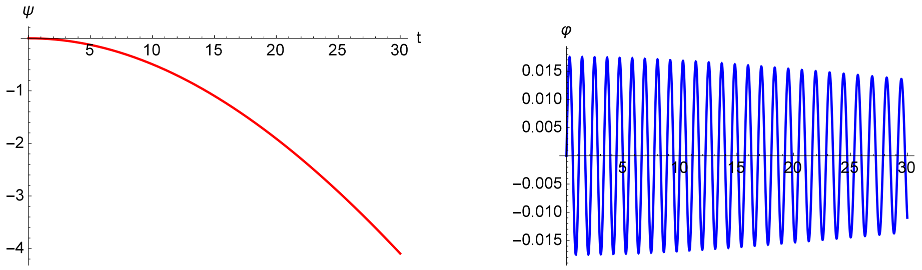

Figure 2: Motion of the swinging Atwood machine in case of $m_1 = m_2$.

One can readily see that in case of absence of oscillations, when $\varphi \equiv 0$, Eq. (4) is satisfied identically, while Eq. (3) takes the form

$$\ddot{\psi} = \frac{gR(m_2 - m_1)}{I_0 + (m_1 + m_2)R^2} \, . \tag{5}$$

Obviously, it determines the uniformly accelerated motion of the system and generalizes Eq. (1) to the case of nonzero mass of the pulley.

2 Main results

Taking into account planar oscillations of the body m_1 complicates the equations of motion significantly (see Eqs. (3)-(4)) and their general solution cannot be found in an analytical form. However, choosing some realistic values of the system parameters, we can find the corresponding numerical solution for different initial conditions and to get some ideas on possible motion of the system.

At first we consider the case of equal masses ($m_1 = m_2$) and assume that the bodies are at rest. As a vertical initial velocity of the bodies induces only their uniformly accelerated motion without oscillations we give the body m_1 a small initial velocity in a horizontal direction. Due to the fact that both bodies have the same mass and are initially in equilibrium, it seems to be quite natural to assume that the system should move in the neighborhood of its equilibrium position. However, this equilibrium position turns out to be unstable and the system moves away from the equilibrium even for very small values of initial velocity.

Actually, solving Eqs. (3)-(4)) with the initial conditions $\psi(0) = \dot{\psi}(0) = \varphi(0) = 0$, $\dot{\varphi}(0) = 0.1$, we obtain a solution shown in Fig. 2. One can readily see that arising oscillation of body m_1 results in a clockwise rotation of the pulley and movement of the bodies in vertical direction. Amplitude of oscillation decreases with time and the term in the right-hand side of Eq. (3) tends to zero as it should be in case of

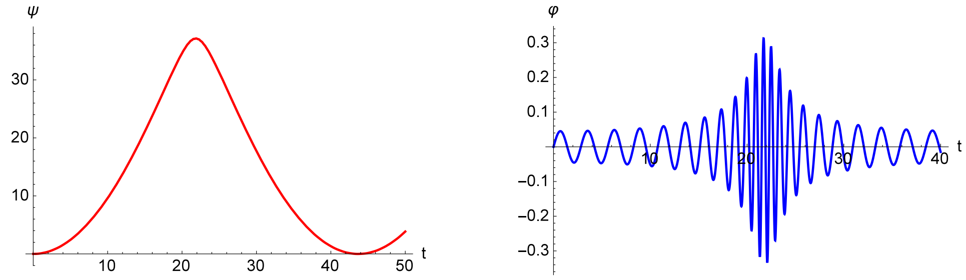

Figure 3: Motion of the swinging Atwood machine in case of $m_1 < m_2$.

the absence of oscillation and equal masses of the bodies. Thus, due to oscillation of the body m_1 a transformation of its initial horizontal momentum into vertical motion of the bodies takes place. A physical reason of such transformation is an increase of a tension of the thread due to oscillation of the body m_1 and appearance of the centrifugal force $m_1 L\dot{\varphi}^2$. Computing an average value of the net force in the right-hand side of Eq. (3) shows that it really becomes smaller than zero when body m_1 oscillates, in spite of the equal masses of the bodies.

Note that the thread tension depends on amplitude of the body m_1 oscillation. If the amplitude is quite small and the mass of body m_2 is greater than mass m_1 the net force in the right-hand side of Eq. (3) may become positive. Then the pulley starts to rotate counterclockwise and the thread length L between the body m_1 and pulley decreases. The amplitude of oscillation and the thread tension starts to grow up and if the masses difference $m_2 - m_1$ is less than some critical value the net force in the right-hand side of Eq. (3) becomes negative again. As a result angular velocity $\dot{\psi}$ of the pulley changes the sign and the system starts to move in opposite direction. Then amplitude of oscillation decreases again and when its value becomes small enough angular velocity $\dot{\psi}$ changes the sign and the process repeats. Thus, we can observe a quasi-periodic motion of the swinging Atwood machine (see Fig. 3). This result is quite unexpected and it should be taken into account when the Atwood machine is used for measuring the gravity acceleration.

3 Conclusions

In the present talk we have analyzed an influence of oscillation on the Atwood machine motion in the simplest case when only one body is permitted to oscillate in a plane. Nevertheless, we have shown that even such oscillation can completely modify a motion of the system, while the simple Atwood machine demonstrates only the uniformly accelerated motion of the bodies. Of course, a mass and size of the pulley and changing the length L between the body m_1 and pulley owing to winding the thread on the pulley affect on the system motion, as well. Doing necessary calculation, we have shown that these factors only change an inertness of the system and modify oscillation of the body m_1 but do not change qualitatively the system behaviour.

It should be noted that there are many physical problems which seem to be quite simple although the corresponding mathematical models are rather complicated to be solved and analyzed by hand. But application of the modern computer algebra systems such as Wolfram Mathematica, for example, helps a lot in analyzing such problems and promotes development of physical intuition and better understanding of the subject.

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