Secular Perturbations of Two Planetary Three-Body Problem with the Masses Changing Anisotropically in Different Rates

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1 Statement of the problem

Let us consider a system of three mutually gravitating spherical celestial bodies T_0, T_1 and T_2 with variable masses

$$m_0 = m_0(t), \quad m_1 = m_1(t), \quad m_2 = m_2(t) \tag{1}$$

changing anisotropically in different rates (law of masses variation is arbitrary) (Refs. [1])

$$\frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_1}{m_1}, \quad \frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_2}{m_2}, \quad \frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2}. \tag{2}$$

On the basis of Meshcherskiy equation (Refs. [2]), we write the equations of motion of three-body problem with variable masses in the presence of reactive forces in the absolute coordinate system

$$m_j\ddot{\vec{R}}_j = grad_{\vec{R}_j} U + \dot{m}_j\vec{V}_j, \quad U = f\left(\frac{m_0 m_1}{R^*_{01}} + \frac{m_0 m_2}{R^*_{02}} + \frac{m_1 m_2}{R^*_{12}}\right),$$

where $\vec{u}_j$are the absolute velocities of the separating particles,

$$\vec{V}_j = \vec{u}_j - \dot{\vec{R}}_j \neq 0, \quad \forall \vec{V}_j \quad j = 0, 1, 2 \tag{3}$$

are the relative velocities of the separating particles, $\vec{R}_j$ are the radius vectors of the center of the spherical bodies, $\vec{R}_{ij}$ are the distances between the centers of the spherical bodies, f is gravitational constant. Following L.G. Lukyanov (Refs. [3]), we assume that the reactive forces are applied to the center of the respective spherical bodies. Usually in the observational astronomy for specific celestial bodies shall be determined the laws of the masses change (see Eqs. (1, 2)) and the relative velocities of the separating particles (see Eqs. (3)). Therefore we assume that the values of (see Eqs. (1, 3)) are known.

It should be noted that in general case of the three-body problem with variable masses changing anisotropically in the different rates is unknown any integral.

Therefore, the problem under consideration is investigated by methods of perturbation theory (Refs. [1], [4, 5, 6]), and with the use of analytical calculations system Mathematica (Refs. [7]).

2 The equations of motion in the Jacobi coordinates

The equations of motion of this problem in the Jacobi coordinates have the form

$$\mu_1 \ddot{\vec{r}}_1 = grad_{\vec{r}_1} U + \vec{F}_1, \quad \mu_2 \ddot{\vec{r}}_2 = grad_{\vec{r}_2} U - (2\dot{\nu}_1 \dot{\vec{r}}_1 + \ddot{\nu}_1 \vec{r}_1) + \vec{F}_2.$$

where reduced masses are of the form

$$\mu_1 = \frac{m_1 m_0}{m_0 + m_1} \neq const, \quad \mu_2 = \frac{m_2(m_0 + m_1)}{m_0 + m_1 + m_2} \neq const, \quad \nu_1 = \frac{m_1}{m_0 + m_1} \neq const.$$

The functions

$$\vec{F}_1 = \vec{F}_1(F_{1x}, F_{1y}, F_{1z}) = \vec{F}_1(t) = \frac{\dot{m}_1}{m_1}\vec{V}_1 - \frac{\dot{m}_0}{m_0}\vec{V}_0 \neq 0,$$

$$\vec{F}_2 = \vec{F}_2(F_{2x}, F_{2y}, F_{2z}) = \vec{F}_2(t) = \left(\frac{\dot{m}_2}{m_2}\vec{V}_2 - \frac{\dot{m}_0}{m_0}\vec{V}_0\right) - \nu_1 \left(\frac{\dot{m}_1}{m_1}\vec{V}_1 - \frac{\dot{m}_0}{m_0}\vec{V}_0\right) \neq 0,$$

are considered known and given.

3 A canonical system of equations of the secular perturbations of two planetary three-body problem with masses changing anisotropically in the analogues of the second system of the Poincaré elements

The equations of motion in osculating elements of the aperiodic motion on the quasi-conical cross-section in the analogues of the Jacobi and Delaunay elements are given in (Refs. [4]).

Furthermore investigation of equations of the secular perturbations is reduced to solving the following system of non-autonomous differential equations

$$\begin{aligned} \dot{\xi}_i &= \frac{\partial R^*_{isec}}{\partial \eta_i}, & \dot{p}_i &= \frac{\partial R^*_{isec}}{\partial q_i}, \\ \dot{\eta}_i &= -\frac{\partial R^*_{isec}}{\partial \xi_i}, & \dot{q}_i &= -\frac{\partial R^*_{isec}}{\partial p_i}. \end{aligned} \tag{4}$$

where R^*_{isec} are perturbation functions (Refs. [4, 5, 6]), ξ_i, η_i, p_i, q_i are analogues of the second system of the Poincaré elements (Refs. [1]).

In this paper in the expansion of the perturbing function are saved the terms up to the second degree inclusive of small quantities $m_1, m_2, e_1, e_2, i_1, i_2$ (Refs. [4, 5, 6]). Thus the secular expressions for R^*_{1sec}, R^*_{2sec} in the analogues of the second system of the Poincaré elements have the form (Refs. [5, 6])

$$\begin{aligned} R^*_{1sec} &= \frac{1}{\gamma_1^2}\cdot\frac{\tilde{\beta}_1^4}{2\mu_{10}\Lambda_1^2} + F_{01} + F_{12sec1} + F_{\rho 1sec} + \Phi_{1sec}, \\ R^*_{2sec} &= \frac{1}{\gamma_2^2}\cdot\frac{\tilde{\beta}_2^4}{2\mu_{20}\Lambda_2^2} + F_{02} + F_{12sec2} + F_{\rho 2sec} + V_{sec} + \Phi_{2sec}, \end{aligned} \tag{5}$$

$$F_{01} = -\frac{b_1\gamma_1^2 a_1^2}{2\psi_1} - f\frac{m_1 m_2}{\gamma_2\psi_1 a_2}, \quad F_{12sec1} = \frac{f}{\psi_1}\left[\frac{m_1 m_2}{r_{12}}\right]_{sec}, \quad F_{\rho 1sec} = -\frac{3b_1\gamma_1^2 a_1^2}{4\Lambda_1\psi_1}(\xi_1^2+\eta_1^2),$$

$$F_{02} = -\frac{b_2\gamma_2^2 a_2^2}{2\psi_2} - f\frac{m_1 m_2}{\gamma_2\psi_2 a_2}, \quad F_{12sec2} = \frac{f}{\psi_2}\left[\frac{m_1 m_2}{r_{12}}\right]_{sec}, \quad F_{\rho 2sec} = -\frac{3b_2\gamma_2^2 a_2^2}{4\Lambda_2\psi_2}(\xi_2^2+\eta_2^2),$$

$$V_{sec} = -\frac{9a_1 a_2\mu_2\gamma_2(2\dot{\gamma}_1\dot{\nu}_1 + \gamma_1\ddot{\nu}_1)}{14\sqrt{\Lambda_1}\sqrt{\Lambda_2}\psi_2}(\xi_1\xi_2 + \eta_1\eta_2),$$

$$\Phi_{1sec} = \frac{3a_1\gamma_1(t)}{2\psi_1\sqrt{\Lambda_1}}\left\{-F_{1x}(t)\xi_1 + F_{1y}(t)\eta_1 + \frac{F_{1z}(t)}{\sqrt{\Lambda_1}}\left[(-\xi_1 q_1 + \eta_1 p_1)\right]\right\},$$

$$\Phi_{2sec} = \frac{3a_2\gamma_2(t)}{2\psi_2\sqrt{\Lambda_2}}\left\{-F_{2x}(t)\xi_2 + F_{2y}(t)\eta_2 + \frac{F_{2z}(t)}{\sqrt{\Lambda_2}}\left[(-\xi_2 q_2 + \eta_2 p_2)\right]\right\}.$$

Analysis of the formulas (see Eqs. (4, 5)) shows that the equations of the secular perturbations in the presence of reactive forces (in the case when masses changing anisotropically) are not split into two systems with respect to the elements ξ_i, η_i and p_i, q_i.

The main purpose of this paper is to identify the explicit form of the equations (see Eqs. (4)) and by Picard method to find their approximate analytical solutions. On the basis of these solutions to obtain an explicit form of the equations of the analogues of the orbital elements.

4 Approximate analytical solutions of the equations of secular perturbations in the analogues of the second system of the Poincaré elements by the method of Picard

We write in explicit form the system of equations (see Eqs. (4))

$$\begin{aligned} \dot{\xi}_1 &= K_5 + K_6 p_1 + 2K_1\eta_1 + K_3\eta_2, \quad \dot{\eta}_1 = K_4 - K_6 q_1 + 2K_1\xi_1 + K_3\xi_2, \\ \dot{\xi}_2 &= K'_5 + K'_6 p_2 + 2K'_2\eta_2 + K'_3\eta_1, \quad \dot{\eta}_2 = K'_4 - K'_6 q_2 + 2K'_2\xi_2 + K'_3\xi_1, \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{p}_1 &= -K_6\xi_1 + 2\psi_1^*(t)\left(\frac{q_1}{\Lambda_1} - \frac{q_2}{\sqrt{\Lambda_1\Lambda_2}}\right), \quad \dot{q}_1 = K_6\eta_1 + 2\psi_1^*(t)\left(\frac{p_1}{\Lambda_1} - \frac{p_2}{\sqrt{\Lambda_1\Lambda_2}}\right), \\ \dot{p}_2 &= -K'_6\xi_2 + 2\psi_2^*(t)\left(\frac{q_2}{\Lambda_2} - \frac{q_1}{\sqrt{\Lambda_1\Lambda_2}}\right), \quad \dot{q}_2 = K'_6\eta_2 + 2\psi_2^*(t)\left(\frac{p_2}{\Lambda_2} - \frac{p_1}{\sqrt{\Lambda_1\Lambda_2}}\right). \end{aligned} \tag{7}$$

In this formulation due to the change of the masses anisotropically and therefore adding of the reactive force appear new values which have the form

$$K_4 = -\frac{3a_1F_{1x}(t)\gamma_1(t)}{2\psi_1\sqrt{\Lambda_1}}, \quad K_5 = \frac{3a_1F_{1y}(t)\gamma_1(t)}{2\psi_1\sqrt{\Lambda_1}} \quad K_6 = \frac{3a_1F_{1z}(t)\gamma_1(t)}{2\psi_1\Lambda_1},$$

$$K_4' = -\frac{3a_2F_{2x}(t)\gamma_2(t)}{2\psi_2\sqrt{\Lambda_2}}, \quad K_5' = \frac{3a_2F_{2y}(t)\gamma_2(t)}{2\psi_2\sqrt{\Lambda_2}} \quad K_6' = \frac{3a_2F_{2z}(t)\gamma_2(t)}{2\psi_2\Lambda_2},$$

and the values K_0, K_1, K_2, K_3, K_0', K_1', K_2', K_3' were obtained in (Refs. [6]).

Using the method of Picard we write the solutions of the equations (see Eqs. (6, 7)) as follows

$$\ni_k(t) = \ni_k(t_0) + \int_{t_0}^{t} \Pi_i^{**}(t, \ni_k(t_0))dt, \tag{8}$$

where $\Pi_i^{**}(t, \ni_k)$ are right sides of the equations (see Eqs. (6, 7)), $\ni_k$ are elements ξ_i, η_i, p_i, q_i and $\ni_{k0} = \ni_k(t_0)$ are their values at the initial time.

The solutions of the equation (see Eqs. (8)) allow to analyze the evolution of the analogues of eccentricities e_i, inclinations i_i, argument of pericenters ω_i and motions of longitude of the ascending nodes Ω_i, longitude of pericenters π_i.

$$e_i^2 = \frac{\ni_{\xi_i}^2 + \ni_{\eta_i}^2}{\Lambda_i}, \quad \sin^2 i_i = \frac{\ni_{p_i}^2 + \ni_{q_i}^2}{\Lambda_i},$$

$$\Omega_i = -arctg\frac{\ni_{q_i}}{\ni_{p_i}}, \quad \pi_i = -arctg\frac{\ni_{\eta_i}}{\ni_{\xi_i}}, \quad \omega_i = \pi_i - \Omega_i, \quad i = 1, 2.$$

It should be noted that all of the calculations have been done with the use of analytical calculations system Mathematica (Refs. [7]).

Conclusion

In the paper is considered the problem of three mutually gravitating spherical celestial bodies with variable masses changing anisotropically in different rates in the general case. A system from eight equations of secular perturbations of the first order is obtained in first time in the analogues of the second system of the Poincaré elements in the presence of reactive forces. Approximate analytical solutions of the equations of secular perturbations in the analogues of the second system of the Poincaré elements are found by the method of Picard. On the basis of these solutions is possible to analyze the evolution of the analogues elements of the orbit.

The results of this paper can be used in the analysis of the dynamical evolution of the triple gravitating systems with masses changing anisotropically in the presence of reactive forces.

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